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INNOVATIONS IN FIXED POINT THEORY ON STRONG CONTROLLED-BRANCIARI-b-DISTANCE WITH AN APPLICATION TO NONLINEAR BOUNDARY PROBLEM

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Abstract. This paper presents the strong-controlled Branciari b-distance, a novel distance proposed to integrate and unify various fixed point results previously documented in the relevant literature. We provide some examples to highlight our findings. In addition, we offer the application within the framework of strong-controlled Branciari b-distance to demonstrate the practicality of the acquired outcomes.

Keywords: Metric space (M_{sp}), Branciari distance (B_d), Strong-Controlled Branciari b-distance (SCB_{bd}) , θ-Branciari contraction (θ_{BC}), and Ciric-Reich-Rus type θ-Branciari contraction ($CRRT - \theta_{BC}$).

1. Introduction

An essential method for demonstrating fixed point findings in M_{sp} is the contraction principle, which Banach first proposed in 1922 [1]. This has resulted in significant growth and enthusiasm in the discipline of fixed point theories, which has broad implications in numerous branches of mathematics. Bakhtin [2] developed $b - M_{sp}$ which expanded the idea of standard M_{sp}, this was further developed by Kamran and Samreen's notion of extended $b - M_{sp}$[3]. Abdelawad then established the concept of controlled $b - M_{sp}$ [4]. This concept was elaborated upon by Santina et al. who progressed it into strong-controlled $b - M_{sp}$ [5]. Nevertheless, Branciari introduced the Branciari distance as a generalization of M_{sp} [6]. It is often referred to as the Branciari metric or the Branciari distance function. Those generalizations open up new avenues and possibilities, forming a vibrant and developing field marked by ongoing research projects [7] [8] [9] [10] [11] [12]. The resulting spaces offer fresh and fascinating interpretations on M_{sp} ideas, indicating potential for a wide range of uses. Moreover, existence and uniqueness issues are fundamentally resolved by fixed point theory, especially when dealing with differential and integral equations. It provides a basic structure for coping with a number of issues, such as integrodifferential equations. Under diverse contraction conditions, such as the θ-contraction presented by Jleli et Samet [13] in the framework of Branciari $M_{sp}, CRRT - \theta_{BC}$ and interpolative- θ_{BC} [14] [15] [16] [17] many researchers have investigated a variety of $b - M_{sp}$. When determining the presence and uniqueness of fixed points for mappings in M_{sp}, the Banach contraction principle-a foundational finding in nonlinear analysis-is essential. Drawing inspiration from Braciari's concepts of B_d and Branciari b-distance, the idea of strong-controlled Branciari b-metric spaces (SCB_{bd}) is introduced in the first section of this article. Moreover, we provide the associated definitions and topological properties in the second section. In the third section, three intriguing principles are introduced: interpolative- $\theta_{BC}, CRRT - \theta_{BC}$, and θ_{BC}. Utilizing these novel contractions, we develop and demonstrate a few fixed point theorems within the context of SCB_{bd}. Additionally, an illustrating example that makes use of different sequences is provided. We offer a solution to a boundary value problem involving a fourth-order differential equation in the last section as an application.

2. Preliminary ASSERTIONS

We start by defining the Branciari distance (B_d) [6], which is one of the ideas put out to broaden and generalize the scope of metric.

Definition 2.1. Let $\mathbb{X}$ be a nonempty set and let $Br : \mathbb{X} \times \mathbb{X} \longrightarrow [0, \infty)$ such that for all $g, h \in \mathbb{X}$ and all $k \neq l \in \mathbb{X} \backslash \{g, h\}$

$$\begin{array}{ll} (Br1) & Br(g,h) = 0 \text{ if and only if } g = h (\text{selfdistance / indistancy}) \\ (Br2) & Br(g,h) = Br(h,g) (\text{symmetry}) \\ (Br3) & Br(g,h) \leq Br(g,k) + Br(k,l) + Br(l,h) \text{ (triangle Theory).} \end{array} \tag{0.1}$$

Then, Br is said to be B_d. Then, the pair $(\mathbb{X}, Br)$ is referred to as B_d space.
Now we revisit the notion of θ-contraction that is established by Jleli and Samet [13].
Consider the set θ that contains all the functions that are continuous and non-decreasing $\vartheta : (0, \infty) \to (1, \infty)$ such that it fulfills the conditions below:
($\square$) for each sequence $\{g_n\} \subset (0, \infty)$, $\quad \lim_{n\to\infty} \vartheta(g_n) = 1 \Leftrightarrow \lim_{n\to\infty} g_n = 0^+$;
($\triangle$) there exist $e \in (0, 1)$ and $L \in (0, \infty]$ such that $\lim_{g\to 0^+} \frac{\vartheta(g)-1}{g^e} = L$.
Several fixed-point results have been improved by using this notion. (see e.g. [19] [20] [21])
Recall the notion of controlled strong $b - M_{sp}$ introduced by Santina et al. [5],

Definition 2.2. Let $\mathbb{X}$ be a non-empty set and let $\omega : \mathbb{X} \times \mathbb{X} \to [1, \infty)$. The following function $SC : \mathbb{X} \times \mathbb{X} \to [0, \infty)$ is called a strong-controlled $b - M_{sp}$ if

(1) $SC(g,h)=0$ iff $g=h$;
(2) $SC(g,h)=SC(h,g)$;
(3) $SC(g,h)\leq SC(g,l)+\omega(l,h)SC(l,h)$,
for all $g,h,l\in\mathbb{X}$. The pair $(\mathbb{X},SC\ )$ is called as a strong-controlled $b-M_{sp}$.
We will merge these two concepts, strong-controlled $b-M_{sp}$ and B_d under the designation of a SCB_{bd} space according to the following:

Definition 2.3. Consider the set $\mathbb{X}$ that contains at least one element and $\omega:\mathbb{X}\times\mathbb{X}\to[1,\infty)$ be a mapping. Hence, the function $SCB_{bd}:\mathbb{X}\times\mathbb{X}\to[0,\infty)$ is said to be a Branciari strong controlled b-measure if it fulfills
(i) $SCB_{bd}(g,h)=0$ if and only if $g=h$;
(ii) $SCB_{bd}(g,h)=SCB_{bd}(h,g)$;
(iii) $SCB_{bd}(g,h)\leq SCB_{bd}(g,l)+SCB_{bd}(l,k)+\omega(k,h)SCB_{bd}(k,h)$,
for all $g,h\in\mathbb{X}$ and all distinct $k,l\in\mathbb{X}\backslash\{g,h\}$. The couple of the symbols $(\mathbb{X},SCB_{bd})$ denotes $SCB_{bd}-M_{sp}$.
Example 2.1. Let $\mathbb{X}=\{10,11,12,13\}$. Define $d_b:\mathbb{X}\times\mathbb{X}\to[0,\infty)$ as follows:

$$\begin{aligned}&d_b(t,t)=0,\forall t\in\mathbb{X},d_b(10,t)=d_b(t,10)=40,\forall t\in\mathbb{X}-\{10\}\\&d_b(11,12)=d_b(12,11)=d_b(11,13)=d_b(13,11)=211\\&d_b(13,12)=d_b(12,13)=999\end{aligned}$$

Consider the symmetric function $\omega:\mathbb{X}\times\mathbb{X}\to[1,\infty)$ with the following characteristics:

$$\begin{aligned}&\omega(t,t)=10,\forall t\in\mathbb{X}\\&\omega(10,11)=3,\omega(10,12)=4,\omega(10,13)=\omega(11,12)=2,\omega(11,13)=9,\omega(12,13)=3\end{aligned}$$

Therefore, $(\mathbb{X},d_b)$ is a $SCB_{bd}-M_{sp}$. In spite of that, we shall observe that
(1) $d_b(12,13)=999>d_b(12,10)+\omega(10,13)d_b(10,13)=120$.
(2) $d_g(12,13)=999>\omega(12,11)d_b(12,11)+\omega(11,10)d_b(11,10)+\omega(10,13)d_b(10,13)=622$.
Thus $(\mathbb{X},d_b)$ is neither a strong controlled metric type space nor a controlled Branciari b-distance space.

Example 2.2. Let $\mathbb{X}=\{1,2,3,4\}$. Define $SCB_{bd}:\mathbb{X}\times\mathbb{X}\to\mathbb{R}$ by $\omega(g,h)=10g+10h+100$ then $(\mathbb{X},SCB_{bd})$ is a strong controlled Branciari b-distance space.
Once all other conditions are met, we shall demonstrate the amended quadrilateral inequality.

$$\begin{aligned}SCB_{bd}(g,h)&=|g-h|^2\\&=|g-l+l-k+k-h|^2\text{ where g}\neq h\neq l\neq k\\&=|g-l|^2+|l-k|^2+|k-h|^2+2|g-l||l-k|\\&+2|l-k||k-h|+2|k-h||g-l|\\&\leq|g-l|^2+|l-k|^2+(10g+10h+100)|k-h|^2\\&=SCB_{bd}(g,l)+SCB_{bd}(l,k)+\omega(g,h)SCB_{bd}(k,h)\end{aligned}$$

Therefore, $SCB_{bd}(g,h)\leq SCB_{bd}(g,l)+SCB_{bd}(l,k)+\omega(g,h)SCB_{bd}(k,h)$
Remark 2.1. If $\omega(g,h)=s=1$, then it is the standard B_d . Since it's widely recognized, b-metric doesn't require continuity. As a result, SCB_{bd} is also not always continuous. We assume that every SCB_{bd} is continuous in the present article.
Now, we present the topological properties of strong controlled Branciari b-distance (SCB_{bd}).

Definition 2.4. Let $\mathbb{X}$ be a set that includes at least one element and endowed with SCB_{bd} and a sequence $\{g_n\}$ in $\mathbb{X}$ is said to be
(a) Convergent to g if for every $\epsilon>0$ there exists $N=N(\epsilon)\in\mathbb{N}$ such that $SCB_{bd}\left(g_n,g\right)<\epsilon$, for all $n\geq N$. Particularly, for this instance, we define $\lim_{n\to\infty}g_n=g$.
(b) Cauchy if for every $\epsilon>0$ there exists $N=N(\epsilon)\in\mathbb{N}$ such that $SCB_{bd}\left(g_m,g_n\right)<\epsilon$, for all $m,n\geq N$.
(c) A SCB_{bd}-metric space $(\mathbb{X},SCB_{bd})$ is said to be complete if every Cauchy sequence in $\mathbb{X}$ is convergent.

3. Primary Findings

We shall commence this portion by providing an introduction to the notion θ-Branciari contraction.
Definition 3.1. Let $(\mathbb{X},SCB_{bd})$ be a $(SCB_{bd})-M_{sp}$ and consider the self-mapping $L:\mathbb{X}\to\mathbb{X}$ where $\mathbb{X}$ is a non-empty set. Then, L is called an θ_{BC} if $\exists\ \vartheta\in\theta$ satisfying

$$\vartheta\left(SCB_{bd}(Lg,Lh)\right)\leq\left[\vartheta\left(SCB_{bd}(g,h)\right)\right]^t\ \text{ if } SCB_{bd}(Lg,Lh)\neq0\text{ for }g,h\in\mathbb{X}.$$

Where $t\in(0,1)$ and $g_p=L^pg_0$ for $g_0\in\mathbb{X}$.

Theorem 3.1. Let $(\mathbb{X},SCB_{bd})$ be a complete $SCB_{bd}-M_{sp}$ and $L:\mathbb{X}\to\mathbb{X}$ be a θ_{BC}. If

$\sup_{s\geq 1}\lim_{i\to+\infty} w\left(g_{2i+2}, g_s\right) < 1$ and for each $g \in \mathbb{X}, \lim_{p\to+\infty} w\left(g, g_p\right)$ exists and is finite,

then L contains only one fixed point in $\mathbb{X}$ according to the following proof.

For any point $g_0 \in \mathbb{X}$ we generate the following iterative sequence $\{g_p\}$ where $g_p = L^p g_0$ for all $p \in \mathbb{N}$
Assume $L^{p_*}g = L^{p_*+1}g$ for some $p_* \in \mathbb{N}$, then $L^{p_*}g$ is certainly a fixed point of L.
Thus, without losing generality, we can presume that $SCB_{bd}\left(L^p g, L^{p+1} g\right) > 0\ \forall\ p \in \mathbb{N}$. From Definition 3.1, we have

$$\begin{aligned}\vartheta\left(SCB_{bd}\left(g_p, g_{p+1}\right)\right) &= \vartheta\left(SCB_{bd}\left(Lg_{p-1}, Lg_p\right)\right)\\ &\leq \left[\vartheta\left(SCB_{bd}\left(g_{p-1}, g_p\right)\right)\right]^t\\ &\leq \left[\vartheta\left(SCB_{bd}\left(g_{p-2}, g_{p-1}\right)\right)\right]^{t^2}\end{aligned}$$

Recursively, we find that

$$\vartheta\left(SCB_{bd}\left(g_p, g_{p+1}\right)\right) \leq \left[\vartheta\left(SCB_{bd}\left(g_0, g_1\right)\right)\right]^{t^p} \tag{3.1}$$

Accordingly, we obtain that

$$1 < \vartheta\left(SCB_{bd}\left(g_p, g_{p+1}\right)\right) \leq \left[\vartheta\left(SCB_{bd}\left(g_0, g_1\right)\right)\right]^{t^p} \text{ for all } p \in \mathbb{N} \tag{3.2}$$

Letting $p \to \infty$ in (3.2), we get $\vartheta\left(SCB_{bd}\left(g_p, g_{p+1}\right)\right) \to 1$ as $p \to \infty$.

From ($\square$), we have

$$\lim_{p\to\infty} SCB_{bd}\left(g_p, g_{p+1}\right) = 0 \tag{3.3}$$

Similarly, we can easily deduce that

$$\lim_{p\to\infty} SCB_{bd}\left(g_p, g_{p+2}\right) = 0 \tag{3.4}$$

From ($\triangle$), there exist $e \in (0,1)$ and $L \in (0,\infty]$ such that

$$\lim_{p\to\infty} \frac{\vartheta\left(SCB_{bd}\left(g_p, g_{p+1}\right)\right) - 1}{\left[SCB_{bd}\left(g_p, g_{p+1}\right)\right]^e} = F$$

Suppose that $F < \infty$. In this case, let $C = \frac{F}{2} > 0$. Utilizing the definition of a limit, choose $p_0 \in \mathbb{N}$ such that

$$\left|\frac{\vartheta\left(SCB_{bd}\left(g_p, g_{p+1}\right)\right) - 1}{\left[SCB_{bd}\left(g_p, g_{p+1}\right)\right]^e} - F\right| \leq C$$

for all $p \geq p_0$.

This implies that $\frac{\vartheta(SCB_{bd}(g_p, g_{p+1})) - 1}{[SCB_{bd}(g_p, g_{p+1})]^e} \geq F - C = C$ for all $p \geq p_0$.

Then, we derive that:

$$p\left[SCB_{bd}\left(g_p, g_{p+1}\right)\right]^e \leq p\left[\frac{\vartheta\left(SCB_{bd}\left(g_p, g_{p+1}\right)\right) - 1}{C}\right] \text{ for all } p \geq p_0$$

Suppose that $F = \infty$. Let $C > 0$ be an arbitrary positive number.

Using the limit definition, we find $p_0 \in \mathbb{N}$ such that $\frac{\vartheta(SCB_{bd}(g_p, g_{p+1})) - 1}{[SCB_{bd}(g_p, g_{p+1})]^e} \geq C$ for all $p \geq p_0$. This implies that

$p\left[SCB_{bd}\left(g_p, g_{p+1}\right)\right]^e \leq p\left[\frac{\vartheta(SCB_{bd}(g_p, g_{p+1})) - 1}{C}\right]$ for all $p \geq p_0$.

Thus, in all cases, there exist $\frac{1}{C} > 0$ and $p_0 \in \mathbb{N}$ such that $p\left[SCB_{bd}\left(g_p, g_{p+1}\right)\right]^e \leq p\left[\frac{\vartheta(SCB_{bd}(g_p, g_{p+1})) - 1}{C}\right]$ for all $p \geq p_0$.

Using equation (3.2), we obtain:

$$p\left[SCB_{bd}\left(g_p, g_{p+1}\right)\right]^e \leq \left[\vartheta\left(SCB_{bd}\left(g_0, g_1\right)\right)\right]^{t^p} - 1 \text{ for all } p \geq p_0$$

Letting $p \to \infty$, then it yields to:

$$\lim_{p\to\infty} p\left[SCB_{bd}\left(g_p, g_{p+1}\right)\right]^e = 0$$

Thus, there exists $p_1 \in \mathbb{N}$ such that

$$SCB_{bd}(g_p, g_{p+1}) \leq \frac{1}{p^{\frac{1}{e}}} \text{ for all } p \geq p_1 \tag{3.5}$$

Let $N = \max\{p_0, p_1\}$. Due to the modified triangle inequality, we have two cases:
Case 1: Let $g_p = g_q$ where $p \neq q$. In case $q > p$, we have $L^{q-p}(g_p) = g_p$. Choose $h = g_p$ and $s = q - p$. Then $L^s h = h$. Consequently, L has h as a periodic point. Hence, $SCB_{bd}(h, Lh) = SCB_{bd}\left(L^s h, L^{s+1}h\right) = SCB_{bd}\left(L^{ks}h, L^{ks+1}h\right)$ for all $k \in \mathbb{N}$. Therefore, it is clear from the above reasoning that $SCB_{bd}(h, Lh) = 0$, so $h = Lh$, that is, h is a fixed point of L.
Case 2: Suppose that $L^p g \neq L^q g$ for all integers $p \neq q$. Let $p < q$ be two natural numbers; to show that $\{g_p\}$ is a Cauchy sequence, we need to consider two subcases:
Subcase 1: We claim that if $p - q$ is odd, then $SCB_{bd}(g_p, g_q)$ converges to 0 as $p, q \to \infty$ To prove this, we may assume that $q = p + 2s + 1$. Thus,

$$\begin{aligned}
SCB_{bd}(g_p, g_{p+2s+1}) &\leq SCB_{bd}(g_p, g_{p+1}) + SCB_{bd}(g_{p+1}, g_{p+2}) + w(g_{p+2}, g_{p+2s+1})\, SCB_{bd}(g_{p+2}, g_{p+2s+1}) \\
&\leq SCB_{bd}(g_p, g_{p+1}) + SCB_{bd}(g_{p+1}, g_{p+2}) + w(g_{p+2}, g_{p+2s+1})\left[SCB_{bd}(g_{p+2}, g_{p+3}) + SCB_{bd}(g_{p+3}, g_{p+4})\right] \\
&+ w(g_{p+4}, g_{p+2s+1})\, SCB_{bd}(g_{p+4}, g_{p+2s+1}) \\
&\leq SCB_{bd}(g_p, g_{p+1}) + SCB_{bd}(g_{p+1}, g_{p+2}) + w(g_{p+2}, g_{p+2s+1})\left[SCB_{bd}(g_{p+2}, g_{p+3}) + SCB_{bd}(g_{p+3}, g_{p+4})\right] \\
&+ w(g_{p+2}, g_{p+2s+1})\, w(g_{p+4}, g_{p+2s+1})\, SCB_{bd}(g_{p+4}, g_{p+2s+1}) \\
&\leq SCB_{bd}(g_p, g_{p+1}) + SCB_{bd}(g_{p+1}, g_{p+2}) + w(g_{p+2}, g_{p+2s+1})\left[SCB_{bd}(g_{p+2}, g_{p+3}) + SCB_{bd}(g_{p+3}, g_{p+4})\right] \\
&+ w(g_{p+2}, g_{p+2s+1})\, w(g_{p+4}, g_{p+2s+1})\left[SCB_{bd}(g_{p+4}, g_{p+5}) + SCB_{bd}(g_{p+5}, g_{p+6})\right] \\
&+ w(g_{p+2}, g_{p+2s+1})\, w(g_{p+4}, g_{p+2s+1})\, w(g_{p+6}, g_{p+2s+1})\, SCB_{bd}(g_{p+6}, g_{p+2s+1}) \\
&\leq SCB_{bd}(g_p, g_{p+1}) + SCB_{bd}(g_{p+1}, g_{p+2}) + w(g_{p+2}, g_{p+2s+1})\left[SCB_{bd}(g_{p+2}, g_{p+3}) + SCB_{bd}(g_{p+3}, g_{p+4})\right] \\
&+ w(g_{p+2}, g_{p+2s+1})\, w(g_{p+4}, g_{p+2s+1})\left[SCB_{bd}(g_{p+4}, g_{p+5}) + SCB_{bd}(g_{p+5}, g_{p+6})\right] \\
&+ w(g_{p+2}, g_{p+2s+1})\, w(g_{p+4}, g_{p+2s+1})\, w(g_{p+6}, g_{p+2s+1})\left[SCB_{bd}(g_{p+6}, g_{p+7}) + SCB_{bd}(g_{p+7}, g_{p+8})\right] \\
&+ w(g_{p+2}, g_{p+2s+1})\, w(g_{p+4}, g_{p+2s+1})\, w(g_{p+6}, g_{p+2s+1})\, w(g_{p+8}, g_{p+2s+1})\, SCB_{bd}(g_{p+8}, g_{p+2s+1}). \\
&\leq \cdots \\
&\leq SCB_{bd}(g_p, g_{p+1}) + SCB_{bd}(g_{p+1}, g_{p+2}) \\
&+ w(g_{p+2}, g_{p+2s+1})\left[SCB_{bd}(g_{p+2}, g_{p+3}) + SCB_{bd}(g_{p+3}, g_{p+4})\right] \\
&+ w(g_{p+2}, g_{p+2s+1})\, w(g_{p+4}, g_{p+2s+1}) \cdots w(g_{p+2s-2}, g_{p+2s+1}) \\
&\left[SCB_{bd}(g_{p+2s-2}, g_{p+2s-1}) + SCB_{bd}(g_{p+2s-1}, g_{p+2s})\right] \\
&+ w(g_{p+2}, g_{p+2s+1})\, w(g_{p+4}, g_{p+2s+1})\, w(g_{p+6}, g_{p+2s+1}) \\
&\cdots w(g_{p+2s}, g_{p+2s+1})\, SCB_{bd}(g_{p+2s}, g_{p+2s+1})
\end{aligned}$$

This can be written as,

$$\begin{aligned}
SCB_{bd}(g_p, g_{p+2s+1}) \leq{}& SCB_{bd}(g_p, g_{p+1}) + SCB_{bd}(g_{p+1}, g_{p+2}) \\
&+ \sum_{i=\frac{p}{2}+1}^{\frac{p+2s-2}{2}} \left(SCB_{bd}(g_{2i}, g_{2i+1}) + SCB_{bd}(g_{2i+1}, g_{2i+2})\right) \prod_{j=\frac{p}{2}+1}^{i} w(g_{2j}, g_{p+2s+1}) \\
&+ \prod_{i=\frac{p}{2}+1}^{\frac{p+2s}{2}} w(g_{2i}, g_{p+2s+1})\, SCB_{bd}(g_{p+2s}, g_{p+2s+1}) \\
\leq{}& SCB_{bd}(g_p, g_{p+1}) + SCB_{bd}(g_{p+1}, g_{p+2}) \\
&+ \sum_{i=\frac{p}{2}+1}^{\frac{p+2s-2}{2}} SCB_{bd}(g_{2i}, g_{2i+1}) \prod_{j=\frac{p}{2}+1}^{i} w(g_{2j}, g_{p+2s+1}) \\
&+ \prod_{i=\frac{p}{2}+1}^{\frac{p+2s}{2}} w(g_{2i}, g_{p+2s+1})\, SCB_{bd}(g_{p+2s}, g_{p+2s+1}) \\
&+ \sum_{i=\frac{p}{2}+1}^{\frac{p+2s-2}{2}} SCB_{bd}(g_{2i+1}, g_{2i+2}) \prod_{j=\frac{p}{2}+1}^{i} w(g_{2j}, g_{p+2s+1})
\end{aligned}$$

Here, we can combine the above inequality as follows:

$$\begin{aligned} SCB_{bd}\left(g_p, g_{p+2s+1}\right) \leq & SCB_{bd}\left(g_p, g_{p+1}\right)+SCB_{bd}\left(g_{p+1}, g_{p+2}\right) \\ & +\sum_{i=\frac{p}{2}+1}^{\frac{p+2s}{2}} SCB_{bd}\left(g_{2i}, g_{2i+1}\right) \prod_{j=\frac{p}{2}+1}^{i} w\left(g_{2j}, g_{p+2s+1}\right) \\ & +\sum_{i=\frac{p}{2}+1}^{\frac{p+2s-2}{2}} SCB_{bd}\left(g_{2i+1}, g_{2i+2}\right) \prod_{j=\frac{p}{2}+1}^{i} w\left(g_{2j}, g_{p+2s+1}\right) \end{aligned}$$

Applying (3.5) to the above inequality yields the following:

$$\begin{aligned} & SCB_{bd}\left(g_p, g_{p+2s+1}\right) \leq \frac{1}{p^{1/e}}+\frac{1}{(p+1)^{\frac{1}{e}}}. \\ & +\sum_{i=\frac{p}{2}+1}^{\frac{p+2s}{2}}\left(\prod_{j=\frac{p}{2}+1}^{i} w\left(g_{2j}, g_{p+2s+1}\right)\right) \frac{1}{(2i)^{1/e}} \\ & +\sum_{i=\frac{p}{2}+1}^{\frac{p+2s-2}{2}}\left(\prod_{j=\frac{p}{2}+1}^{i} w\left(g_{2j}, g_{p+2s+1}\right)\right) \frac{1}{(2i+1)^{1/e}} \end{aligned}$$

Now, let

$$L_c=\sum_{i=1}^{c}\left(\prod_{j=1}^{i} w\left(g_{2j}, g_{p+2s+1}\right)\right) \frac{1}{(2i)^{1/e}}.$$

and

$$N_d=\sum_{i=1}^{d}\left(\prod_{j=1}^{i} w\left(g_{2j}, g_{p+2s+1}\right)\right) \frac{1}{(2i+1)^{1/e}}.$$

Then, we obtain:

$$SCB_{bd}\left(g_p, g_{p+2s+1}\right) \leq \frac{1}{p^{1/2}}+\frac{1}{(p+1)^{1/e}}+\left[L_{\frac{p+2s}{2}}-L_{\frac{p}{2}}\right]+\left[N_{\frac{p+2s-2}{2}}-N_{\frac{p}{2}}\right]$$

By using the fact $w\left(g_p, g_q\right)<1$ and applying the ratio test for convergence of series, we can deduce that

$$\lim_{p, s \rightarrow \infty}\left[L_{\frac{p+2s}{2}}-L_{\frac{p}{2}}\right]=0 \text{ and } \lim_{p, s \rightarrow \infty}\left[N_{\frac{p+2s-2}{2}}-N_{\frac{p}{2}}\right]=0$$

Moreover,

$$\lim_{p \rightarrow \infty}\left[\frac{1}{p^{\frac{1}{e}}}\right]=0$$

and

$$\lim_{p \rightarrow \infty}\left[\frac{1}{(p+1)^{\frac{1}{e}}}\right]=0, \text{ where } e \in(0,1).$$

Therefore,

$$\lim_{p, s \rightarrow \infty} SCB_{bd}(g_p, g_{p+2s+1})=0$$

Subcase 2: We claim that if $p-q$ is even, then $SCB_{bd}(g_p, g_q)$ converges to a positive number as $p, q \to \infty$ To prove this we may assume that $q = p+2s$. Thus,

$$\begin{aligned}
SCB_{bd}(g_p, g_{p+2s}) &\leq SCB_{bd}(g_p, g_{p+1}) + SCB_{bd}(g_{p+1}, g_{p+2}) + w(g_{p+2}, g_{p+2s})\, SCB_{bd}(g_{p+2}, g_{p+2s}) \\
&\leq SCB_{bd}(g_p, g_{p+1}) + SCB_{bd}(g_{p+1}, g_{p+2}) + w(g_{p+2}, g_{p+2s})\left[SCB_{bd}(g_{p+2}, g_{p+3}) + SCB_{bd}(g_{p+3}, g_{p+4})\right] \\
&+ w(g_{p+2}, g_{p+3})\, w(g_{p+4}, g_{p+2s})\, SCB_{bd}(g_{p+4}, g_{p+2s}) \\
&\leq SCB_{bd}(g_p, g_{p+1}) + SCB_{bd}(g_{p+1}, g_{p+2}) + w(g_{p+2}, g_{p+2s})\left[SCB_{bd}(g_{p+2}, g_{p+3}) + SCB_{bd}(g_{p+3}, g_{p+4})\right] \\
&+ w(g_{p+2}, g_{p+3})\, w(g_{p+4}, g_{p+2s})\left[SCB_{bd}(g_{p+4}, g_{p+5}) + SCB_{bd}(g_{p+5}, g_{p+6})\right] \\
&+ w(g_{p+2}, g_{p+3})\, w(g_{p+4}, g_{p+2s})\, w(g_{p+6}, g_{p+2s})\, SCB_{bd}(g_{p+6}, g_{p+2s}) \\
&\leq \cdots \\
&\leq SCB_{bd}(g_p, g_{p+1}) + SCB_{bd}(g_{p+1}, g_{p+2}) \\
&+ w(g_{p+2}, g_{p+2s})\left[SCB_{bd}(g_{p+2}, g_{p+3}) + SCB_{bd}(g_{p+3}, g_{p+4})\right] \\
&+ w(g_{p+2}, g_{p+3})\, w(g_{p+4}, g_{p+2s}) \cdots w(g_{p+2s-3}, g_{p+2s}) \\
&\left[SCB_{bd}(g_{p+2s-3}, g_{p+2s-2}) + SCB_{bd}(g_{p+2s-2}, g_{p+2s-1})\right] \\
&+ w(g_{p+3}, g_{p+2s})\, w(g_{p+5}, g_{p+2s})\, w(g_{p+7}, g_{p+2s}) \\
&\cdots w(g_{p+2s-1}, g_{p+2s})\, SCB_{bd}(g_{p+2s-1}, g_{p+2s})
\end{aligned}$$

This can be written as,

$$\begin{aligned}
SCB_{bd}(g_p, g_q) &\leq SCB_{bd}(g_{p+2}, g_{p+1}) + SCB_{bd}(g_{p+1}, g_{p+2}) \\
&+ \sum_{i=\frac{p}{2}+1}^{\frac{p+2s-3}{2}} \left(SCB_{bd}(g_{2i}, g_{2i+1}) + SCB_{bd}(g_{2i+1}, g_{2i+2})\right) \prod_{j=\frac{p}{2}+1}^{i} w(g_{2j}, g_{p+2s}) \\
&+ \prod_{i=\frac{p}{2}+1}^{\frac{p+2s-1}{2}} w(g_{2i}, g_{p+2s})\, SCB_{bd}(g_{p+2s-1}, g_{p+2s})
\end{aligned}$$

$$\begin{aligned}
SCB_{bd}(g_p, g_{p+2s}) \leq & SCB_{bd}(g_p, g_{p+1}) + SCB_{bd}(g_{p+1}, g_{p+2}) \\
&+ \sum_{i=\frac{p}{2}+1}^{\frac{p+2s-3}{2}} SCB_{bd}(g_{2i}, g_{2i+1}) \prod_{j=\frac{p}{2}+1}^{i} w(g_{2j}, g_{p+2s}) \\
&+ \prod_{i=\frac{p}{2}+1}^{\frac{p+2s-1}{2}} w(g_{2i}, g_{p+2s})\, SCB_{bd}(g_{p+2s-1}, g_{p+2s}) \\
&+ \sum_{i=\frac{p}{2}+1}^{\frac{p+2s-3}{2}} SCB_{bd}(g_{2i+1}, g_{2i+2}) \prod_{j=\frac{p}{2}+1}^{i} w(g_{2j}, g_{p+2s}) \\
\leq & SCB_{bd}(g_p, g_{p+1}) + SCB_{bd}(g_{p+1}, g_{p+2}) \\
&+ \sum_{i=\frac{p}{2}+1}^{\frac{p+2s-1}{2}} SCB_{bd}(g_{2i}, g_{2i+1}) \prod_{j=\frac{p}{2}+1}^{i} w(g_{2j}, g_{p+2s}) \\
&+ \sum_{i=\frac{p}{2}+1}^{\frac{p+2s-3}{2}} SCB_{bd}(g_{2i+1}, g_{2i+2}) \prod_{j=\frac{p}{2}+1}^{i} w(g_{2j}, g_{p+2s})
\end{aligned}$$

Again, employing (3.5), we reach the following:

$$\begin{aligned}
SCB_{bd}(g_p, g_{p+2s}) \leq & \frac{1}{p^{1/e}} + \frac{1}{(p+1)^{1/e}} + \sum_{i=\frac{p}{2}+1}^{\frac{p+2s-1}{2}} \frac{1}{(2i)^{1/e}} \prod_{j=\frac{p}{2}+1}^{i} w(g_{2j}, g_{p+2s}) \\
&+ \sum_{i=\frac{p}{2}+1}^{\frac{p+2s-3}{2}} \frac{1}{(2i+1)^{\frac{1}{e}}} \prod_{j=\frac{p}{2}+1}^{i} w(g_{2j}, g_{p+2s})
\end{aligned}$$

Now, let us consider the following substitutions to simplify the above inequality.

Let $L_c = \sum_{i=1}^{c} \left(\prod_{j=1}^{i} w(g_{2j}, g_{p+2s})\right) \frac{1}{(2i)^{1/e}}$

and $N_d = \sum_{i=1}^{d} \prod_{j=1}^{i} w\left(g_{2j}, g_{p+2s}\right) \frac{1}{(2i+1)^{\frac{1}{e}}}$

Then, the simplified inequality will be as follows:

$$SCB_{bd}\left(g_p, g_{p+2s}\right) \leq \frac{1}{p^{1/e}} + \frac{1}{(p+1)^{\frac{1}{e}}} + \left[L_{\frac{p+2s}{2}} - L_{\frac{p}{2}}\right] + \left[N_{\frac{p+2s-2}{2}} - N_{\frac{p}{2}}\right]$$

Using a similar procedure as in the previous part in correlation with the ratio test, we prove the following:

$$\lim_{p,s\to\infty} SCB_{bd}(g_p, g_{p+2s}) = 0$$

Therefore, $\{g_p\}$ is Cauchy sequence in $\mathbb{X}$.
Given that $(\mathbb{X}, SCB_{bd})$ is a complete SCB_{bd}, it implies that the sequence $\{g_p\}$ converges to a point μ in $\mathbb{X}$.
Following this, we shall show the continuity of L. Assume that if $Lg \neq Lh$ and by employing (3.1), we obtain

$$\begin{aligned} \ln\left[\vartheta SCB_{bd}(Lg, Lh)\right] &\leq t \ln\left[\vartheta SCB_{bd}(g, h)\right] \\ &\leq \ln\left[\vartheta SCB_{bd}(g, h)\right] \end{aligned}$$

Given that ϑ is non-decreasing, the aforementioned observance leads to the conclusion that $SCB_{bd}(Lg, Lh) \leq SCB_{bd}(g, h)$ for all distinct $g, h \in \mathbb{X}$.
From this inspection, we can get, $SCB_{bd}\left(g_{p+1}, L\mu\right) = SCB_{bd}\left(Lg_p, L\mu\right) \leq SCB_{bd}\left(g_p, \mu\right) \forall\ p \in \mathbb{N}$.
If we take $p \to \infty$ in the preceding inequality, we obtain $g_{p+1} \to L\mu$.
Utilizing axiom 3 in definition 2.3, we get

$$SCB_{bd}(\mu, L\mu) \leq SCB_{bd}\left(\mu, g_p\right) + SCB_{bd}\left(g_p, g_{p+1}\right) + \omega\left(g_{p+1}, L\mu\right) SCB_{bd}\left(g_{p+1}, L\mu\right)$$

Letting $p \to \infty$, applying (3.5) and the definition 1.1 of (ii), we get $SCB_{bd}(\mu, L\mu) = 0$, this means $L\mu = \mu$,
That opposes the argument that L lacks a periodic point. Then, consider period e and take μ to be a periodic point of L. Assume that L has a phi set of fixed points.
Hence, $SCB_{bd}(z, Lz) > 0\ \forall\ z \in \mathbb{X}$ and $SCB_{bd}\left(z, L^e z\right) = 0$ for $e > 1$
Using Definition 2.1, we get

$$\begin{aligned} \vartheta\left(SCB_{bd}(z, Lz)\right) &= \vartheta\left(SCB_{bd}\left(L^e z, L^{e+1} z\right)\right) \\ &\leq \left[\vartheta\left(SCB_{bd}(z, Lz)\right)\right]^{t^e} \\ &< \vartheta\left(SCB_{bd}(z, Lz)\right) \end{aligned}$$

Therefore, this results in a contradiction. Hence, $\exists$ a point $\mu \in \mathbb{X}$ where $L\mu = \mu$.
Assume that $\xi \in L$ is a fixed point different than μ. Accordingly, $SCB_{bd}(\mu, \xi) = SCB_{bd}(L\mu, L\xi) \neq 0$.
Now using the condition (3.1), we get,

$$\begin{aligned} \vartheta\left(SCB_{bd}(\mu, \xi)\right) = \vartheta\left(SCB_{bd}(L\mu, L\xi)\right) &= \vartheta\left(SCB_{bd}\left(L^e \mu, L^e \xi\right)\right) \\ &\leq \left[\vartheta\left(SCB_{bd}(\mu, \xi)\right)\right]^{r^e} \\ &< \vartheta\left(SCB_{bd}(\mu, \xi)\right), \text{ a contradiction.} \end{aligned}$$

Therefore, $\mu = \xi$. Consequently, L is asserted to possess one particular fixed point in $\mathbb{X}$.
Now, let us consider the following example validate our findings. Example 3.1.
Construct the following sequence:

$$\begin{aligned} \sigma_1 &= 1 \times 2 \\ \sigma_2 &= 1 \times 2 + 2 \times 5 \\ \sigma_3 &= 1 \times 2 + 2 \times 5 + 3 \times 10 \\ \sigma_4 &= 1 \times 2 + 2 \times 5 + 3 \times 10 + 4 \times 17 \\ \sigma_p &= 1 \times 2 + 2 \times 5 + 3 \times 10 + 4 \times 17 + \ldots. + p\left(p^2 + 1\right) = p^3 + p \\ &= \left[\frac{p(p+1)}{2}\right]^2 + \frac{p(p+1)}{2} = \frac{p^4 + 2p^3 + 3p^2 + 2p}{4} \end{aligned} \tag{3.6}$$

Let $g = \{\sigma_p : p \in \mathbb{N}\}$. Define $SCB_{bd} : \mathbb{X} \times \mathbb{X} \to [0, \infty)$ as $SCB_{bd}(g, h) = |g - h|^2$. Consider$w : \mathbb{X} \times \mathbb{X} \to [1, \infty)$ as $\omega(g, h) = 5g + 3h + 20$. Subsequently, $(\mathbb{X}, SCB_{bd})$ is a complete SCB_{bd} space.
Let $L : \mathbb{X} \to \mathbb{X}$ be a self-mapping such that $L\left(\sigma_1\right) = \sigma_1, L\left(\sigma_p\right) = \sigma_{p-1}\ \forall\ p \geq 2$. We will now demonstrate that L is a θ_{BC} where $\vartheta(x) = e^x$.
Since $\theta\left(SCB_{bd}(Lg, Lh)\right) \leq \left[\vartheta\left(SCB_{bd}(g, h)\right)\right]^t$ which yields $e^{(SCB_{bd}(Lg,Lh))} \leq \left[e^{(SCB_{bd}(g,h))}\right]^t$. Applying log on both sides we get,

$$SCB_{bd}(Lg, Lh) \leq tSCB_{bd}(g, h)$$

Therefore, proving the preceding equation is sufficient to demonstrate that L is a θ_{BC}.

Case-1: If $p = 1$ and $q > 2$, then

$$\begin{aligned}
SCB_{bd}\left(L\sigma_1, L\sigma_q\right) &= SCB_{bd}\left(\sigma_1, \sigma_{q-1}\right) \\
&= \left|\frac{(q-1)^4 + 2(q-1)^3 + 3(q-1)^2 + 2(q-1) - 8}{4}\right|^2 \\
&= \left|\frac{(q-1)\left[(q-1)^3 + 2(q-1)^2 + 3(q-1) + 2\right] - 8}{4}\right|^2 \\
&= \left|\frac{(q-1)\left[q^3 - 3q^2 + 3q - 1 + 2q^2 - 4q + 2 + 3q - 3 + 2\right] - 8}{4}\right|^2 \\
&= \left|\frac{(q-1)\left[q^3 - q^2 + 2q\right] - 8}{4}\right|^2 \\
&= \left|\frac{q^4 - 2q^3 + 3q^2 - 2q - 8}{4}\right|^2
\end{aligned}$$

and

$$SCB_{bd}\left(\sigma_1, \sigma_q\right) = \left|\frac{q^4 + 2q^3 + 3q^2 + 2q - 8}{4}\right|^2$$

Now consider,

$$\begin{aligned}
\frac{SCB_{bd}\left(L\sigma_1, L\sigma_q\right)}{SCB_{bd}\left(\sigma_1, \sigma_q\right)} &= \left|\frac{q^4 - 2q^3 + 3q^2 - 2q - 8}{q^4 + 2q^3 + 3q^2 + 2q - 8}\right|^2 \\
&< t \text{ given that } t \in (0, 1).
\end{aligned}$$

Case-2: For $q > p > 1$, we have

$$\begin{aligned}
SCB_{bd}\left(L\sigma_p, L\sigma_q\right) &= SCB_{bd}\left(\sigma_{p-1}, \sigma_{q-1}\right) \\
&= SCB_{bd}\left(\frac{(p-1)^4 + 2(p-1)^3 + 3(p-1)^2 + 2(p-1)}{4}, \frac{(q-1)^4 + 2(q-1)^3 + 3(q-1)^2 + 2(q-1)}{4}\right) \\
&= \left|\frac{(p-1)^4 + 2(p-1)^3 + 3(p-1)^2 + 2(p-1)}{4} - \frac{(q-1)^4 + 2(q-1)^3 + 3(q-1)^2 + 2(q-1)}{4}\right| \\
&= \left|\frac{(p-1)^4 - (q-1)^4 + 2\left[(p-1)^3 - (q-1)^3\right]}{4} + \frac{3\left[(p-1)^2 - (q-1)^2\right] + 2(p-q)}{4}\right|^2
\end{aligned}$$

and

$$\begin{aligned}
SCB_{bd}\left(\sigma_p, \sigma_q\right) &= SCB_{bd}\left(\frac{p^4 + 2p^3 + 3p^2 + 2p}{4}, \frac{q^4 + 2q^3 + 3q^2 + 2q}{4}\right) \\
&= \left|\frac{p^4 + 2p^3 + 3p^2 + 2p}{4} - \frac{q^4 + 2q^3 + 3q^2 + 2q}{4}\right|^2 \\
&= \left|\frac{\left(p^4 - q^4\right) + 2\left(p^3 - q^3\right) + 3\left(p^2 - q^2\right) + 2(p-q)}{4}\right|^2
\end{aligned}$$

Now consider,

$$\begin{aligned}
\frac{SCB_{bd}\left(L\sigma_p, L\sigma_q\right)}{SCB_{bd}\left(\sigma_p, \sigma_q\right)} &= \left|\frac{(p-1)^4 - (q-1)^4 + 2\left[(p-1)^3 - (q-1)^3\right] + 3\left[(p-1)^2 - (q-1)^2\right] + 2(p-q)}{\left(p^4 - q^4\right) + 2\left(p^3 - q^3\right) + 3\left(p^2 - q^2\right) + 2(p-q)}\right|^2 \\
&< t \text{ given that } t \in (0, 1).
\end{aligned}$$

Hence, L meets θ_{BC} with $\vartheta(x)=e^x$. Then, from Theorem 2.1, L has one particular fixed point σ_1.
Letting $w(g,h)=1$ in the preceding theorem, the following corollary is obtained.
Corollary 3.1.Consider L as a self mapping on a complete B_d space. If $\exists\ \Theta\in\theta$ and $t\in(0,1)$ satisfying

$$\Theta(d(Lg,Lh))\leq[\Theta(d(g,h))]^t \text{ when } d(Lg,Lh)\neq 0 \text{ for } g,h\in\mathbb{X}$$

then L possesses one particular fixed point in $\mathbb{X}$.

Definition 3.2. Consider the $SCB_{bd}-M_{sp}$ $(\mathbb{X},SCB_{bd})$. The self-mapping $\mathfrak{L}:\mathbb{X}\to\mathbb{X}$ is said to be Ćirić-Reich-Rus type θ_{BC}, briefly, $CRRT-\theta_{BC}$, if there exists a function $\vartheta\in\theta$ and non-negative real number $t<1$ such that

$$\vartheta\left(SCB_{bd}(\mathfrak{L}g,\mathfrak{L}h)\right)\leq\left[M_{\mathfrak{L},\vartheta}(g,h)\right]^t \tag{3.7}$$

for all $g,h\in\mathbb{X}$, where

$$M_{\mathfrak{L},\vartheta}(g,h):=\max\left\{\vartheta\left(SCB_{bd}(g,h)\right),\vartheta\left(SCB_{bd}(h,\mathfrak{L}h)\right)\vartheta\left(SCB_{bd}(g,\mathfrak{L}g)\right)\right\}$$

where $\limsup_{p,q\to\infty}\omega\left(g_p,g_q\right)<1$, here $g_p=f^pg_0$ for $g_0\in\mathbb{X}$ and $t\in(0,1)$.

Theorem 3.2. Consider $(\mathbb{X},SCB_{bd})$ a complete SCB_{bd} space and $\mathfrak{L}:\mathbb{X}\to\mathbb{X}$ be a $CRRT-\theta_{BC}$. Then $\mathfrak{L}$ has one particular fixed point.
As in Theorem 3.1, we establish an iterative sequence $\{g_p\}_{0\infty}$. Let $g_0\in\mathbb{X}$ and define:

$$g_p=\mathfrak{L}^pg_0\forall\ p\in\mathbb{N}$$

Keeping generality intact, we presume that $SCB_{bd}\left(\mathfrak{L}^pg,\mathfrak{L}^{p+1}g\right)>0$ for all $p\in\mathbb{N}$. Certainly, if $\mathfrak{L}^{p_*}g=\mathfrak{L}^{p_*+1}g$ for some $p_*\in\mathbb{N}$, then $\mathfrak{L}^{p_*}g$ will be a fixed point of L.
We show that $\lim_{p\to\infty}SCB_{bd}\left(g_p,g_{p+1}\right)=0$.
Applying the condition of contraction (3.7), we obtain,

$$\vartheta\left(SCB_{bd}\left(g_{p+1},g_p\right)\right)\leq\left[M_{\mathfrak{L},\vartheta}\left(g_p,g_{p-1}\right)\right]^t \tag{3.8}$$

in which

$$\begin{aligned}M_{\mathfrak{L},\vartheta}\left(g_p,g_{p-1}\right)&=\max\left\{\vartheta\left(SCB_{bd}\left(g_p,g_{p-1}\right)\right),\vartheta\left(SCB_{bd}\left(g_p,\mathfrak{L}g_p\right)\right),\vartheta\left(SCB_{bd}\left(g_{p-1},\mathfrak{L}g_{p-1}\right)\right)\right\}\\&=\max\left\{\vartheta\left(SCB_{bd}\left(g_p,g_{p-1}\right)\right),\vartheta\left(SCB_{bd}\left(g_p,g_{p+1}\right)\right),\vartheta\left(SCB_{bd}\left(g_{p-1},g_p\right)\right)\right\}\\&\leq\max\left\{\vartheta\left(SCB_{bd}\left(g_p,g_{p-1}\right)\right),\vartheta\left(SCB_{bd}\left(g_p,g_{p+1}\right)\right)\right\}\end{aligned}$$

If $M_{\mathfrak{L},\theta}\left(g_p,g_{p-1}\right)=\vartheta\left(SCB_{bd}\left(g_p,g_{p+1}\right)\right)$, afterwards the inequality (3.8) turns into $\theta\left(SCB_{bd}\left(g_{p+1},g_p\right)\right)\leq\vartheta\left(SCB_{bd}\left(g_p,g_{p+1}\right)\right)^t\Leftrightarrow\ln\left(\vartheta\left(SCB_{bd}\left(g_{p+1},g_p\right)\right)\right)\leq t\ln\left(\vartheta\left(SCB_{bd}\left(g_{p+1},g_p\right)\right)\right)$, which is a contradiction (because $t<1$). Hence, we have $M_{\mathfrak{L},\vartheta}\left(g_p,g_{p-1}\right)=\vartheta\left(SCB_{bd}\left(g_{p-1},g_p\right)\right)$. From (3.8), it follows that

$$\vartheta\left(SCB_{bd}\left(g_p,g_{p+1}\right)\right)\leq\left[\vartheta\left(SCB_{bd}\left(g_{p-1},g_p\right)\right)\right]^t$$

Repeatedly, we discover that

$$\vartheta\left(SCB_{bd}\left(g_p,g_{p+1}\right)\right)\leq\left[\vartheta\left(SCB_{bd}\left(g_0,g_1\right)\right)\right]^{t^p}$$

Following this insight, we deduce that $\{g_p\}$ in $\mathbb{X}$ is Cauchy sequence by tracing the relevant lines in the Theorem 3.2 proof.
In conclusion, the sequence $\{g_p\}$ in $\mathbb{X}$ is Cauchy. Because, $(\mathbb{X},SCB_{bd})$ is a complete SCB_{bd}, there is a certain point μ in $\mathbb{X}$ in a way that $\{g_p\}$ converges to μ.
Keeping generality intact, we presume that $\mathfrak{L}^pg\neq\mu$ for all p (or p tends to infinity). Assume that $d(\mu,L\mu)>0$. Using (3.7), we obtain

$$\vartheta\left(SCB_{bd}\left(\mathfrak{L}g_p,\mathfrak{L}\mu\right)\right)\leq\left[M_{\mathfrak{L},\vartheta}\left(g_p,\mu\right)\right]^t \tag{3.9}$$

for all $g,h\in\mathbb{X}$, in which

$$M_{\mathfrak{L},\vartheta}\left(g_p,\mu\right):=\max\left\{\vartheta\left(SCB_{bd}\left(g_p,\mu\right)\right),\vartheta\left(SCB_{bd}\left(g_p,\mathfrak{L}g_p\right)\right),\vartheta\left(SCB_{bd}(\mu,\mathfrak{L}\mu)\right)\right\}$$

Taking $p\to\infty$ in the preceding inequality, we get

$$\vartheta\left(SCB_{bd}(\mu,\mathfrak{L}\mu)\right)\leq\left[\vartheta\left(SCB_{bd}(\mu,\mathfrak{L}\mu)\right)\right]^t<\vartheta\left(SCB_{bd}(\mu,\mathfrak{L}\mu)\right)$$

that is a contradiction. Therefore, $\mathfrak{L}\mu=\mu$.
Hence, $\mathfrak{L}$ contains a fixed point in $\mathbb{X}$.

Assume that $\mu\neq\xi$ are distinct fixed points of $\mathfrak{L}$. Afterwards, obviously $SCB_{bd}(\mu,\xi)=SCB_{bd}(\mathfrak{L}\mu,L\xi)\neq 0$.
Applying condition (3.10) now, we obtain,

$$\begin{aligned}1 < \vartheta\left(SCB_{bd}(\mu,\xi)\right) &= \vartheta\left(SCB_{bd}(\mathfrak{L}\mu,\mathfrak{L}\xi)\right)\\ &\leq \left[\max\left\{\vartheta\left(SCB_{bd}(\mu,\xi)\right),\vartheta\left(SCB_{bd}(\mu,\mathfrak{L}\mu)\right),\vartheta\left(SCB_{bd}(\xi,\mathfrak{L}\xi)\right)\right\}\right]^{t}\\ &< \vartheta\left(SCB_{bd}(\mu,\xi)\right)\end{aligned}$$

that is clearly a contradiction. Consequently, we obtain $\xi = \mu$.
As a result, $\mathfrak{L}$ has just one fixed point in $\mathbb{X}$.

Definition 3.3. Consider the SCB_{bd} space, $(\mathbb{X}, SCB_{bd})$ and a self-mapping $\mathfrak{L}:\mathbb{X}\to\mathbb{X}$. Hence, $\mathfrak{L}$ is considered an interpolative- θ_{BC} when $\exists$ a function $\vartheta\in\theta$ such that $t_1+t_2+t_3<1$ where t_1,t_2,t_3 are positive real numbers satisfying

$$\begin{aligned}\vartheta\left(SCB_{bd}(\mathfrak{L}g,\mathfrak{L}h)\right) \leq{}& \left[\vartheta\left(SCB_{bd}(g,h)\right)\right]^{t_1}\left[\vartheta\left(SCB_{bd}(g,\mathfrak{L}g)\right)\right]^{t_2}\\ &\left[\vartheta\left(SCB_{bd}(h,\mathfrak{L}h)\right)\right]^{t_3},\end{aligned} \tag{3.10}$$

for all $g,h\in\mathbb{X}$, where $\limsup_{p,q\to\infty}\omega\left(g_p,g_q\right)<1$, here $g_p=\mathfrak{L}^p g_0$ for $g_0\in\mathbb{X}$ and $t\in(0,1)$.

Theorem 3.3. Let $(\mathbb{X}, SCB_{bd})$ be a complete and continuous function SCB_{bd}. If $\mathfrak{L}:\mathbb{X}\to\mathbb{X}$ is an interpolative- θ_{BC}, then in $\mathbb{X},\mathfrak{L}$ has a single fixed point.
We omit this proof because

$$\begin{aligned}&\left[\vartheta\left(SCB_{bd}(g,h)\right)\right]^{t_1}\left[\theta\left(SCB_{bd}(g,\mathfrak{L}g)\right)\right]^{t_2}\left[\vartheta\left(SCB_{bd}(h,\mathfrak{L}h)\right)\right]^{t_3}\\ &\qquad\leq\left[M_{\vartheta,\mathfrak{L}}(g,h)\right]^{t_1+t_2+t_3}\end{aligned}$$

Then, selecting $t:=t_1+t_2+t_3<1$ is sufficient in Theorem 3.2 to sum up the preceding Theorem.

4. Application to Differential Equations

Consider the following system of differential equations:

$$\begin{cases}\psi^4(\chi)=g\left(\chi,\psi(\chi),\psi',\psi'',\psi'''\right)\\ \psi(0)=\psi'(0)=\psi''(1)=\psi'''(1)=0;\chi\in[0,1]\end{cases} \tag{4.1}$$

so that g is a continuous function defined as $g:[0,1]\times\mathbb{R}^3\times\mathbb{R}\to\mathbb{R}$.

The focus of the study is on the boundary value problem (BVP), a fundamental concept in mathematical analysis, particularly when applied to the modeling of complex physical phenomena. In this case, the problem is situated within the context of elastic beam deformations, with an emphasis on the equilibrium configuration. Specifically, the problem models scenarios where one end of the beam is free to move, while the other is fixed in place. This setup is commonly referred to as the cantilever beam problem in the field of mechanics, underlining its wide applicability and importance in both theoretical and applied mathematics.
The solutions to this type of BVP play a crucial role in numerous applications, ranging from structural engineering to mathematical physics. As such, determining the existence and uniqueness of solutions is of central importance. To achieve this, we apply the fixed-point technique, a well-established method in mathematical analysis that is particularly effective for solving BVPs of this nature.
Here we examine if a boundary value problem with a differential equation of fourth order has a solution. On the interval $[0,1]$, $\mathbb{X}=\mathcal{F}[0,1]$ is the notation for the space of continuous bounded functions. Within this space, we introduce the SCB_{bd} as a means to measure distances between functions. The metric is denoted by:

$$SCB_{bd}(L(v),g(v))=\max_{v\in\mathbb{X}}|L(v)-g(v)|^2$$

which provides the necessary structure for analyzing the convergence of sequences of functions in $\mathbb{X}$.
Furthermore, we define a mapping $\omega:\mathbb{X}^4\to[1,\infty)$, where $\omega\left(u_1,u_2,u_3,u_4\right)=5u_1+5u_2+3$. This mapping introduces an additional layer of complexity to our exploration, enabling us to further study the behavior of solutions within the context of the BVP.
Through this approach, we aim to demonstrate the existence of a solution to the boundary value problem, using the fixed-point theorem and the introduced metrics to rigorously establish the
conditions under which solutions can be found. Having established the foundational framework, we now proceed to reformulate the fourth-order ordinary differential equation (BVP) as an integral equation. The integral form in which the BVP is presented as below:

$$\psi(t)=\int_0^1\mathcal{G}(\chi,v)g\left(v,\psi(v),\psi'(v)\right)dv,\quad \psi\in\mathcal{F}[0,1]$$

In this expression, $\mathcal{G}(\chi,v)$ represents Green's function associated with the homogeneous linear problem:

$$\psi^{(4)}(\chi)=0, \quad \psi(0)=\psi'(0)=\psi''(1)=\psi^{(3)}(1)=0$$

This function provides a key component for solving the boundary value problem and offers valuable insight into the underlying structure of the equation. The use of Green's function allows for a detailed understanding of the interactions between the boundary conditions and the solution, facilitating the analysis of the problem's characteristics.

$$\mathcal{G}(\chi,v)=\begin{cases}\frac{1}{6}\chi^2(3v-\chi), & 0\le\chi\le v\le 1\\ \frac{1}{6}v^2(3\chi-v), & 0\le v\le\chi\le 1\end{cases} \tag{4.2}$$

The subsequently properties of $\mathcal{G}(\chi,v)$ could be simply verified from (3.2).

$$\frac{1}{3}\chi^2v^2\le\mathcal{G}(\chi,v)\le\frac{1}{2}\chi^2\left(\text{ or }\frac{1}{2}v^2\right),\chi,v\in[0,1]$$

Theorem 4.1. Assume that the subsequent constraints are satisfied.
(1) The mapping $g:[0,1]\times\mathbb{R}^3\times\mathbb{R}\to\mathbb{R}$ is continuous.
(2) For every $\psi,y\in\mathbb{X}$, there is a $\sigma\in[1,\infty)$ such that the below condition is satisfied.

$$\left|g\left(v,\psi,\psi'\right)-g\left(v,y,y'\right)\right|\le\sqrt{20}e^{\frac{-\sigma}{2}}|\psi(v)-y(v)|, \quad v\in[0,1]$$

(3) There exists $\psi_0\in\mathbb{X}$ where for every $\chi\in[0,1]$, we deduce

$$\psi_0t\le\int_0^1\mathcal{G}(\chi,v)g\left(v,\psi_0(v),\psi_0'(v)\right)dv$$

Hence, the BVP problem contains a solution in $\mathbb{X}$.

Proof: Assume the self-mapping $L:\mathbb{X}\to\mathbb{X}$ is defined as

$$L(\psi)(\chi)=\int_0^1\mathcal{G}(\chi,v)g\left(v,\psi(v),\psi'(v)\right)dv$$

Then $\psi=L\psi$, which imply that the BVP has only one solution.

Consider,

$$\begin{aligned}|L(\psi)(\chi)-L(y)(\chi)|^2&=\left|\int_0^1\mathcal{G}(\chi,v)g\left(v,\psi(v),\psi'(v)\right)dv-\int_0^1\mathcal{G}(\chi,v)g\left(v,y(v),y'(v)\right)dv\right|^2\\&\le\int_0^1(\mathcal{G}(\chi,v))^2\mid g\left(v,\psi(v),\psi'(v)\right)-g\left(v,y(v),\,y'(v)\right|^2dv\\&\le\int_0^1\frac{1}{4}v^420e^{-\sigma}|\psi(v)-y(v)|^2dv\\&\le 20e^{-\sigma}SCB_{bd}(\psi,y)\int_0^1\frac{1}{4}v^4dv\\&\le 20e^{-\sigma}SCB_{bd}(\psi,y)\frac{1}{20}\\&=e^{-\sigma}SCB_{bd}(\psi,y)\end{aligned}$$

where this deduce to,

$$SCB_{bd}(L(\psi),L(y))\le e^{-\sigma}SCB_{bd}(\psi,y)$$
$$\sqrt{SCB_{bd}(L(\psi),L(y))}\le\sqrt{e^{-\sigma}SCB_{bd}(\psi,y)}$$

$$e^{\sqrt{SCB_{bd}(L(\psi),L(y))}}\le\left(e^{\sqrt{SCB_{bd}(\psi,y)}}\right)^{\sqrt{e^{-\sigma}}}\text{; where }e^{-\sigma}<1\text{ as }\sigma\ge1$$

Thus, $e^{\sqrt{SCB_{bd}(L(\psi),L(y))}}\le\left(e^{\sqrt{SCB_{bd}(\psi,y)}}\right)^{\sqrt{r}}$ with $r=\sqrt{e^{-\sigma}}$ which gives,

$$\theta\left(SCB_{bd}(L\psi,Ly)\right)\le\left[\theta\left(SCB_{bd}(\psi,y)\right)\right]^r\text{ where }\theta(t)=e^{\sqrt{\chi}}$$

L has a fixed point as all conditions of Theorem 3.1 are met. Therefore, the solution to BVP is found in $\mathbb{X}$.

5. Conclusion

As an extension of strong controlled metric type space and Branciari space, we introduced the notion of strong controlled b-Branciari metric type space in this work. Next, in the framework of strong controlled-b-Branciari metric type space, we proved several fixed point theorems concerning Theta-contraction, Ciric-Reich-Rus type Theta-Brainciari contraction, and interpolative-Theta-Branciari contraction. Additionally, we provide numerous examples to highlight our conclusions.

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